Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 1320A – The first midterm exam **Instructor: K. Zaynullin**

Last name: Sivasubramaniam.

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Signature:

Student number: 6782706.

Read the following instructions:

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Question	1	2	3	4	5	6	Total
Mark	3	1	3	4	2	1	14
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1. The following is a table of some values of two functions $y = f(x)$ and $y = g(x)$.

x	1	2	3	4
$f(x)$	2	3	1	1
$g(x)$	1	4	3	2

3

Find

(a) $(f \circ g)(2) = \boxed{1}$ (1)

(b) $(g \circ f)(1) = \boxed{4}$ (1)

(c) $(g \circ g)(3) = \boxed{3}$ (1)

f(g(2)

g(2) = 4

g(f(1)

g(2)

g(g(3)

g(3)

2. Consider the function

$$f(x) = \frac{2\ln(x)}{\ln(x) - 1}$$

1

(a) What is the domain of the function f? (1)

The domain is $\boxed{\{x \mid x > 0,\ x \in \mathbb{R}\}}$ ✗ need $x \neq e$

(b) Find the inverse of the function f. (2)

The inverse fucntion is $\boxed{y = -e^{x} - 1}$ ✗

(c) What is the range of the function f? (1)

The range of f is $\boxed{\{y \mid y \in \mathbb{R}\}}$ ✗

$y = \dfrac{2\ln x}{\ln(x-1)}$

$\ln y = x$

$y = e^{x}$

$x = \dfrac{2\ln y}{\ln(y-1)}$

$y = \dfrac{2\ln x}{\ln x - 1}$

$\ln x = y$

$e^{y} = \dfrac{2x}{x-1}$

$\times\dfrac{\ln x + 1}{\ln x + 1}$

$y = \dfrac{2\ln x}{\ln x - 1}$

Some right ideas ✓

$e^{y} = \dfrac{2x}{x-1}$

$e^{x} = \dfrac{2y}{y-1}$

$e^{y} = \dfrac{2x}{x-1}$

$y - 1\, e^{x} = 2y$

$e^{x} = \dfrac{2y}{y-1}$

$y = \dfrac{y-1\, e^{x}}{2}$

$e^{x}(y-1) = 2y$

$2y = y - e^{x} - 1$

$\dfrac{ye^{x} - e^{x}}{y}$ ~~2y~~ 2

$y = -e^{x} - 1$

$ye^{x} = 2y + e^{x}$

$y = \dfrac{2y + e^{x}}{e^{x}\ \cancel{2y}}$

$$y = \frac{2\ln x}{\ln(x) - 1}$$

$$y = \frac{2\ln x}{\ln(x)-1} \times \frac{\ln(x)+1}{\ln(x)+1}$$

$$y = \frac{2\ln x(\ln(x)+1)}{\ln(x)^2 - 1}$$

$$y = \frac{2\ln(x)^2 + 2\ln x}{\ln x^2 - 1}$$

$$y = \frac{2\ln x(\cancel{\ln(x)+1})}{(\ln x - 1)(\cancel{\ln(x+1)})}$$

$$y = \frac{2\ln(x)}{\ln(x)-1}$$

$$(\ln(x)-1)\,y = \frac{2\ln x}{y}$$

$$\ln(x) = y$$

$$e^y = x$$

$$x = \cancel{\frac{2\ln y}{\ln(y)-1}}$$

$$y = \frac{2\ln x}{\ln(x)-1}$$

$$y(\ln(x)-1) = 2\ln x$$

$$y\ln(x) - y = 2\ln x$$

$$y\ln(x) - 2\ln x = y$$

$$\ln x(y-2) = y$$

$$\ln x = \frac{y}{y-2}$$

$$\ln x = y$$

$$x = e^y$$

$$x = e^{\frac{y}{y-2}}$$

$$y = e^{\frac{x}{x-2}}$$

3. Exponential and logarithmic functions:

3

(a) What is $\log_3 \frac{1}{\sqrt{3}}$? (1)

The answer is $\boxed{-\frac{1}{2}}$

(b) Solve for x, if $\log_2(x+1) + \log_2(x-1) = 2$. (2)

The answer $x \doteq \boxed{2.23}$ (2.227834407)

$\log_2(x+1) + \log_2(x-1) = 2.$

$\log_2 (x+1)(x-1) = 2.$

$\log_2(x^2-1) = 2.$

$\log_2(x+1) + \log_2(x-1) = 2.$

$\log_2(x+1)(x-1) = 2$

$\log_2(x^2-1) = 2.$

$3.3.$

$\log_2(5) + \log_2(1)$

$\frac{\ln 5}{\ln 2} + \frac{\ln 1}{\ln 2}$

$\log x = y$ $\quad \log x = \log 2(4)$

$y = \log x \quad \log x = 1.204$

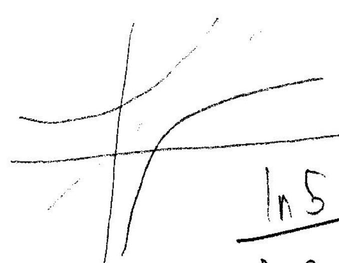
$x = e^y \quad x = e^{1.204}.$

Try to write more clearly

$\log_3 \frac{1}{\sqrt{3}}$

$\log_a x = y$

$\frac{\ln(\frac{1}{\sqrt{3}})}{\ln(3)}.$



$\frac{\ln 5}{\ln 2} + \frac{\ln 1}{\ln 2} = 2.$

$\frac{\ln 5 + \ln 1}{\ln 2} = 2.$

$\frac{\log a x}{\log a x} = \frac{y}{\log a x}.$

$\frac{\ln x}{\ln a} = y$

$\ln x = \ln a y.$

$x = ay.$

$\log a x$

$\frac{\log x}{\log a} = y.$

$\log x = \log a(y).$

4. Finding limits:

(a) (1)

$$\lim_{x\to\infty}\frac{2x+5}{x-4} = \boxed{2.}$$

(b) (1)

$$\lim_{x\to 4^+}\ln(x^2-16) = \boxed{-\infty}$$

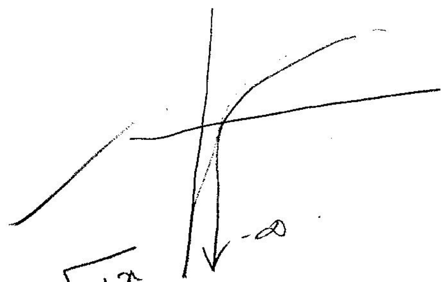
(c) (2)

$$\lim_{x\to 0}\frac{2-\sqrt{4+x}}{x} = \boxed{-1/4}$$

$\lim_{x\to\infty} \frac{2x+5}{x-4}. \quad \frac{2}{1}$

$\lim_{x\to 4^+} \ln(x^2-16)$

$\ln(16-16)$



Right technique

$\frac{2-\sqrt{4+x}}{x} \times \frac{2+\sqrt{4+x}}{2+\sqrt{4+x}}$

$\frac{\cancel{4}-\cancel{4}-\cancel{x}\,-1}{\cancel{x}(2+\sqrt{4+x})}$

$\frac{4-(4+x)}{x(2+\sqrt{4+x}}$

$\frac{-1}{2+\sqrt{4+x}}$

$\frac{-1}{2+\sqrt{4}}$

$\frac{-1}{4}$

5. Use the definition of the derivative to find the derivative of the function

(a) $y = x^2 + x$ when $x = 1$. (1)

Answer $\boxed{3}$

2

(b) $y = x^3$ when $x = -1$. (1)

Answer $\boxed{3}$

Show your computations here:

a. $y = x^2 + x$

$f(1) = 1^2 + 1$

$= 2$.

$\frac{dy}{dx} = \lim_{h \to 0} \frac{[(x+h)^2 + (x+h)] - (x^2+x)}{h}$

$= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 + x + h - (x^2+x)}{h}$

$= \lim_{h \to 0} \frac{1^2 + 2h + h^2 + 1 + h - (2)}{h}$

$= \lim_{h\to 0} \frac{\cancel{2} + 3h + h^2 - \cancel{2}}{h}$

$= \lim_{h \to 0} \frac{h^2 + 3h}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(h+3)}{\cancel{h}}$

$= \lim_{h \to 0} h + 3$

$= 0 + 3$

$= 3$.

b. $y = x^3$

$f(-1) = -1$

$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$

$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - \cancel{x}^3}{h}$

$= \lim_{h\to 0} \frac{(-1)^3 + 3(-1)^2h + 3(-1)h^2 + h^3 + 1}{h}$

$= \lim_{h \to 0} \frac{\cancel{-1} + 3h - 3h^2 + h^3 + \cancel{1}}{h}$

$= \lim_{h\to 0} \frac{h^3 - 3h^2 + 3h}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(h^2 - 3h + 3)}{\cancel{h}}$

$= \lim_{h\to 0} h^2 - 3h + 3$

$= 0 - 3(0) + 3$

$= 3$

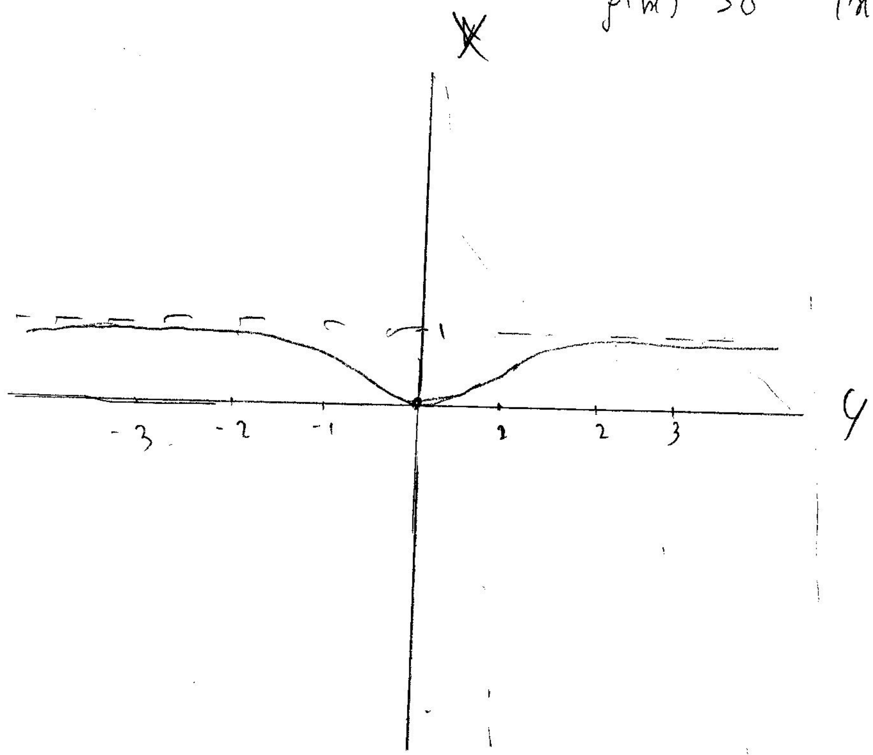
6. Consider a continuous function $y = f(x)$ defined for all real values of x. Suppose this function satisfies all of the following conditions:

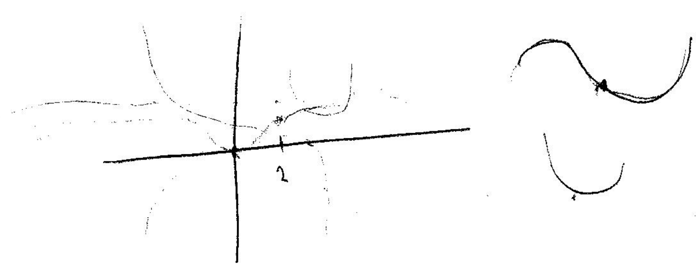
- $f'(x) > 0$ if $|x| < 2$, $f'(x) < 0$ if $|x| > 2$,
- $f'(2) = 0$, $\lim_{x \to \infty} f(x) = 1$, $f(-x) = -f(x)$,
- $f''(x) < 0$ if $0 < x < 3$, $f''(x) > 0$ if $x > 3$.

Sketch the graph of this function. (4)

$f'(x) > 0 \quad |x| < 2$.

1





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1. Find derivatives of the following functions

(a) $y = 2^{\sin \pi x}$. Answer $y' = \boxed{(\sin \pi x \ln 2)(x + \sin \pi)}$ (1)

(b) $y = \arctan(x^2 - 1)$. Answer $y' = \boxed{\frac{2}{x-2}}$ (1)

(c) $y = x^{\cos(x)}$. Answer $y' = \boxed{\frac{e^{-\sin^2 x \ln^2 x} + e^{\cos x}}{e^x}}$ (2)

Show your work here

a. $y = 2^{\sin \pi x}$

$= (\sin \pi x \ln 2)(x + \sin \pi)$

$= x \sin \pi x \ln 2 + \sin^2 \pi x \ln 2$

b. $y = \tan^{-1}(x^2 - 1)$

$= \frac{1}{1 + (x^2-1)^2} \cdot (2x)$

$= \frac{2x}{1 + x^2 - 2x + 1}$

$= \frac{2x}{x^2 - 2x}$

$= \frac{2x}{x(x-2)}$

$= \frac{2}{x-2}$

c. $y = x^{(\cos x)}$

$\ln y = \ln x^{\cos x}$

$\ln y = \cos x \ln x$

$\ln y = (-\sin x)(\ln x) + (\cos x)(\frac{1}{x})$

$\ln y = \frac{-\sin x \ln x}{1} + \frac{\cos x}{x}$

$\ln y = \frac{-\sin^2 x \ln^2 x + \cos x}{x}$

$y = \frac{e^{-\sin^2 x \ln^2 x} + e^{\cos x}}{e^x}$

2. Use implicit differentiation to find an equation of the tangent line to the curve $sin(x + y) = 2x - 2y$ at the point (π, π). (3)

The tangent line is given by the equation $y = \boxed{0.67\pi + 1.036}$ X

2

Show your work here

$\sin(x+y) = 2x - 2y \quad (\pi, \pi)$

$\cos(x+y) \cdot y' = 2 - 2y'$

$y' \cos(x+y) = 2 - 2y'$

$\therefore y' = \dfrac{2 - 2y'}{\cos(x+y)}$

$3y' = \dfrac{2}{\cos(x+y)}$

$y' = \dfrac{2}{3\cos(x+y)}$

Almost!

$y' = \dfrac{2}{3\cos(2\pi)}$

$m = \cancel{2} \quad 0.67$

$y = mx + b$

$\pi = 0.67\pi + b$

$b = 1.036$

$\therefore y = 0.67\pi + 1.036$

$2 \times 3 + 2$

$\cos(a+b) = \cos a \sin$

$= \cos a +$

$= \sin a +$

1.999

0.998

3. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after 3 seconds. (2)

Answer. The rate is $\boxed{67858.40\ cm^2/s}$

2

Show your work here

$\frac{dr}{dt} = 60cm/s.$

$\frac{60cm}{s} \times 3/s$

after 3 s $= ? \frac{da}{dt}$

area $= \pi r^2$

$2\pi r \cdot t'$

$\frac{da}{dt} = \frac{da}{dr} \times \frac{dr}{dt}$

$r = 60$

$s = 1$

$s = 3$

$r = 60 \times 3$

r at 3 sec $= \frac{60cm}{\cancel{sec}} \times 3\cancel{sec}$

$= 180 cm.$

area $= \pi r^2$

$\frac{da}{dt} = \frac{da}{dr} = 2\pi r$

$\frac{da}{dt} = 2\pi r \times \frac{60 cm}{s}$

$\frac{da}{dt} = 2 \times \pi \times 180cm \times \frac{60 cm}{s}$

$= \frac{67858.4.0132 cm^2}{s} \times \frac{1 m^2}{100 cm^2}$

$= 678.58 m/s.$

4. **Find** the absolute maximum and absolute minimum values of the function $f(x) = x - \ln(x)$ on the interval $[\frac{1}{2}, 2]$. (2)

Answer. The values are

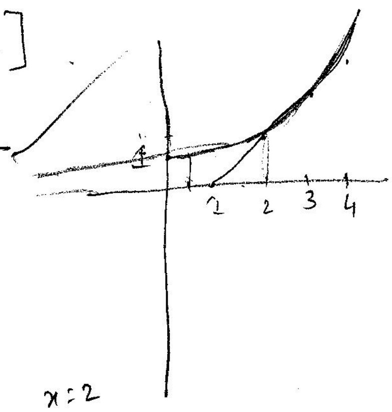
absolute min = (1,1)
absolute max = (2,1.3)

2

Show your work here

$f(x) = x - \ln(x) \qquad [\frac{1}{2}, 2]$

$f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$



$x = \frac{1}{2}, 1, 1.5, 2$

	$x<\frac{1}{2}$	$x=\frac{1}{2}$ $\frac{1}{2}<x<1$	$x=1$ $1<x<1.5$	$x=1.5$ $1.5<x<2$	$x=2$ $x>2$
$\frac{x-1}{x}$	$-$	$-$	$+$	$+$	$+$
	↘	↘	↗	↗	↗

absolute min = $(1, f(1))$
$= (1, 1)$

absolute max = $(2, f(2))$
$= (2, 1.3)$

5. Given $f(x) = \ln(x^4 + 27)$ find

5

(a) the local maximum and minimum values of f, (3)

Answer: local minimum at $(0, 3.3)$
no local maximum.

3

(b) the inflection points and the intervals of concavity. (3)

Answer: Intervals of concavity: $x<0$ and $0<x<3$ (concaving down)
inflection point: $(3, 4.7)$

2

Show your work here

$f(x) = \ln(x^4 + 27)$

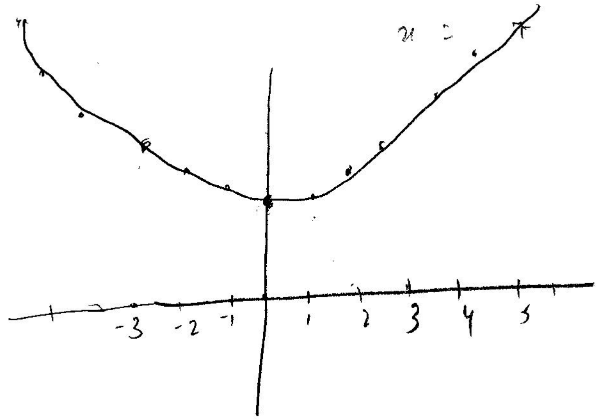
$f'(x) = \frac{1}{x^4+27} \times (4x^3)$

$0 = \frac{4x^3}{x^4+27}$

$x^4 = 0$

$x^4 = -27$

$x =$



$x = 0$

$x=0$

	$x<0$	$x>0$
$4x^3$	$-$	$+$
x^4+27	$+$	$+$
$f'(x)$	$-$ ↘	$+$ ↗

$\therefore$ local minimum at $(0, f(0))$

$= (0, 3.3)$

$f''(x) = \frac{12x^2(x^4+27) - 4x^3(4x^3)}{(x^4+27)^2}$

$= \frac{12x^6 + 324x^2 - 16x^6}{(x^4+27)^2}$

$= \frac{-4x^6 + 324x^2}{(x^4+27)^2}$

→ continued at the back.

5

$$0 = \cancel{-4}\ \frac{-4x^2(x^4-81)}{(x^4+27)^2}$$

$-4x^2 = 0$

$x^4 - 81$

$x^4 = 81$

$x^4 + 27 = 0.$

$x^4 = 27 -$

$x = 0, 3$

	$x<0$	$0<x<3$	$x>3$
$-4x^2$	$-$	$-$	$-$
(x^4-81)	$-$	$-$	$+$
$(x^4+27)^2$	$+$	$+$	$+$
$f'(x)$	$+$	$+$	$-$
	$\cap$	$\cap$	$\cup$

$\therefore$ the function is concave ~~up~~ down at the intervals $x<0$ and $0<x<3$.

The point of inflection is

$(3, f(3))$

$= (3, 4.7)$

~~6~~. Using l'Hospital Rule find the limits

(a)

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \boxed{0}$$

1

(1)

(b)

$$\lim_{x\to\infty} x\tan(1/x) = \boxed{0.0174.}$$

(2)

Show your work here

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}$$

$$= \lim_{x\to\infty} \frac{1}{x} \div \frac{1}{2\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{2\sqrt{x}}{x}$$

$$= \infty.$$

$$\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} = 0.0174$$

$\frac{1}{2\sqrt{x}}$

0.174